Deterministic Hashing to Elliptic and Hyperelliptic Curves

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Elliptic curve cryptography

- $F$ finite field of characteristic $> 3$ (for simplicity’s sake).
- Recall that an elliptic curve over $F$ is the set of points $(x, y) \in F^2$ such that:
  \[ y^2 = x^3 + ax + b \]
  (with $a, b \in F$ fixed parameters), together with a point at infinity.
- This set of points forms an abelian group where the Discrete Logarithm Problem and Diffie-Hellman-type problems are believed to be hard (no attack better than the generic ones).
- Interesting for cryptography: for $k$ bits of security, one can use elliptic curve groups of order $\approx 2^{2k}$, keys of length $\approx 2k$. Also come with rich structures such as pairings.
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Hashing to elliptic curves is a problem

- Many cryptographic protocols (schemes for encryption, signature, PAKE, IBE, etc.) involve representing a certain numeric value, often a hash value, as an element of the group $\mathbb{G}$ where the computations occur.
- For $\mathbb{G} = \mathbb{Z}_n^*$, simply taking the numeric value itself mod $n$ is usually appropriate.
- However, if $\mathbb{G}$ is an elliptic curve group, this technique has no obvious counterpart; e.g. one cannot put the value in the $x$-coordinate of a curve point, because only about $1/2$ of possible $x$-values correspond to actual points.
- Elliptic curve-specific protocols have been developed to circumvent this problem (ECDSA for signature, Menezes-Vanstone for encryption, ECMQV for key agreement, etc.), but doing so with all imaginable protocols is unrealistic.
The traditional solution

- For $k$ bits of security:
  1. concatenate the hash value with a counter from 0 to $k - 1$;
  2. initialize the counter as 0;
  3. if the concatenated value is a valid $x$-coordinate on the curve, i.e. $x^3 + ax + b$ is a square in $F$, return one of the two corresponding points; otherwise increment the counter and try again.

- Heuristically, the probability of a concatenated value being valid is 1/2, so $k$ iterations ensure $k$ bits of security.
Problems with this solution

- A natural implementation does not run in constant time: possible timing attacks (especially for PAKE).
- A constant time implementation (always do $k$ steps, compute the Legendre symbol in constant time) is very inefficient, $O(n^4)$.
- Security is difficult to analyze.

Remark: hashing as $H(m) = h(m)G$ where $G$ is a generator of the elliptic curve group is not a good idea.
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Supersingular curves

An elliptic curve shape of particular interest is:

\[ y^2 = x^3 + b \]

over a field with \( q \) elements, with \( q \equiv 2 \pmod{3} \).
Admits the following deterministic encoding:

\[ f : u \mapsto ((u^2 - b)^{1/3}, u) \]

Such a curve is supersingular. Convenient for pairings, but much less secure than ordinary curves for the same key size (because of the MOV attack).
Shallue-Woestijne-Ulas


Based on Skałba’s identity: if \( g(x) = x^3 + ax + b \), there are rational functions \( X_i(t) \) such that

\[
g(X_1(t)) \cdot g(X_2(t)) \cdot g(X_3(t)) = X_4(t)^2
\]

Hence, on a finite field, at least one of \( g(X_1(t)), g(X_2(t)), g(X_3(t)) \) is a square.

Gives a deterministic point construction algorithm, which is efficient if \( q \equiv 3 \pmod{4} \). Considered for implementation in European e-passports.
Particularly simple deterministic encoding on ordinary elliptic curves when $q \equiv 2 \pmod{3}$, presented by Icart at CRYPTO last year. Generalization of the supersingular case.

Defined as $f : u \mapsto (x, y)$ with

$$x = \left( v^2 - b - \frac{u^6}{27} \right)^{1/3} + \frac{u^2}{3} \quad y = ux + v \quad v = \frac{3a - u^4}{6u}$$

This simple idea sparked new research into the subject of deterministic hashing into elliptic curves.
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Questions we solved

The previous constructions do not completely address the problem of constructing “good hash functions” to elliptic curves, and open up a series of related questions.

We solved some of them.

- **Icart’s conjecture**: Icart observed that his function did not map to the whole elliptic curve, and conjectured that the image comprised only about 5/8 of all points. Is this true? What about the SWU function?

- In particular if $f$ is Icart’s function and $h$ is a random oracle into the base field, $m \mapsto f(h(m))$ is easily distinguished from a random oracle. Can $f$ still be used to construct a random oracle to the curve?

- **Extension to hyperelliptic curves**: can we construct good hash functions? Note that we should map to the Jacobian variety, not the curve itself!
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$E$ elliptic curve over $\mathbb{F}_q$, with $q \equiv 2 \pmod{3}$, and $f : \mathbb{F}_q \rightarrow E(\mathbb{F}_q)$ Icart’s deterministic encoding.

Conjecture (Icart)

There exists a universal constant $C$ such that:

$$\left| \#f(\mathbb{F}_q) - \frac{5}{8} \#E(\mathbb{F}_q) \right| \leq C \sqrt{q}$$

Icart’s paper presented a heuristic argument to justify the constant $5/8$. The conjecture was proved independently by Farahashi, Shparlinski and Voloch, and by Fouque and T.

A consequence of this conjecture is that $f$ is neither injective nor surjective. However, $(u, v) \mapsto f(u) + f(v)$ is a surjective encoding function for $q$ large enough.
Proof idea 1

• A key fact is that $u$ maps to $(x, y)$ under $f$ if and only if:

$$u^4 - 6xu^2 + 6yu - 3a = 0$$

• Hence, the problem is to count the points $(x, y)$ on the curve such that the polynomial $P(u) = u^4 - 6xu^2 + 6yu - 3a$ has at least one root in $\mathbb{F}_q$.

• $P$ can be seen as a polynomial over the function field $\mathbb{F}_q(x, y)$ of $E$, and the problem is to count places of degree 1 in this function field where the reduction of $P$ has a root.

• Mathematicians have a powerful tool to tackle this kind of problems: the Chebotarev density theorem, which says that the “density” of places at which $P$ reduces into a product of factors of given degrees is determined by the number of permutations with the corresponding cycle decomposition in the Galois group of $P$. 
Proof idea II

At this point, completing the proof is a technical exercise:

- Show that $P$ is an irreducible polynomial with Galois group $S_4$ (hard part).
- Count the number of permutations in $S_4$ with a fixed point (there are $1 + 6 + 8 = 15$ of them).
- Deduce that the density of places in $\mathbb{F}_q(x, y)$ at which $P$ has a root is $15/24 = 5/8$.
- Apply an effective version of Chebotarev’s density theorem to get the same result with a $O(\sqrt{q})$ error term for places of degree 1 (this gives Icart’s conjecture).

In the paper with Fouque, we also show how the technique generalizes to other encoding functions with different Galois groups such as a simplified version of SWU (Galois group $D_8$, constant $3/8$).
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Since Icart’s function $f$ (or SWU, etc.) only covers a limited fraction of points on the curve, $m \mapsto f(h(m))$ is not a well-behaved hash function: easy to distinguish from a random oracle.

While some schemes may not require randomness or collision resistance, it is desirable in general to have a construction indistinguishable from a random oracle, in the ROM for some $\mathbb{F}_q$-valued hash function $h$.

Coron and Icart showed it suffices to have an encoding function $F : S \rightarrow E(\mathbb{F}_q)$ from some set $S$, such that $F^{-1}$ is efficiently computable, and that if $s$ is uniformly distributed in $S$, the distribution of $F(s)$ is statistically indistinguishable from uniform in $E(\mathbb{F}_q)$. 
Admissible encodings

- An encoding verifying the statistical indistinguishability property is called *admissible* by Coron and Icart (generalization of a previous definition by Boneh-Franklin).

- Using Maurer’s indifferentiability framework, they show that if $F$ is admissible, then $m \mapsto F(h(m))$ can be used as a random oracle in the ROM for $h$.

- An example of such an admissible encoding is $F(u, v) = f(u) + v \cdot G$ with $G$ a generator of the elliptic curve group. The addition of $vG$ acts as a “one-time pad” to mask the irregularities of $f$, and ensure statistical indistinguishability. Hence

$$H(m) = f(h_1(m)) + h_2(m) \cdot G$$

is a “good” hash function. Also works with SWU, with characteristic 2 counterparts, etc. However, the multiplication makes it slow.
Efficient indifferentiable hashing with Icart

- Since the “easy” admissible encoding is slow, we proposed the following much more efficient solution:

\[ F(u, v) = f(u) + f(v) \]

- We know as a corollary of Icart’s conjecture that this is surjective, but we can also prove statistical indistinguishability with some algebraic geometry machinery.

- Basic idea: for some given point \( \varpi \) on \( E \), the set of \( (u, v) \) in the affine plane such that \( F(u, v) = \varpi \) forms an algebraic curve of bounded genus, that will usually be irreducible.

- In that case, the Hasse-Weil bound ensures that:

\[ F^{-1}(\varpi) = q + O(\sqrt{q}) \]

giving admissibility.

- Making the idea work involves beautiful algebraic geometry (such as intersection theory on the surface \( C \times C \), where \( C \) is the quartic covering of \( E \) defined by the polynomial \( P \) from the previous section).
Efficient indifferentiable hashing, general case

- Previous geometric method: works well for Icart’s function, but difficult to generalize (for SWU, multiple components with complicated interplay; in higher genus, simply horrible).

- We recently proposed a much simpler technique based on character sums. We call an encoding \( f : \mathbb{F}_q \to E(\mathbb{F}_q) \) well-distributed when for any nontrivial character \( \chi \) of \( E(\mathbb{F}_q) \):

\[
\left| \sum_{u \in \mathbb{F}_q} \chi(f(u)) \right| \leq B \sqrt{q}
\]

- Completely formal to show that if \( f \) is well-distributed, \((u, v) \mapsto f(u) + f(v)\) is admissible: write down the statistical distance.

- Relatively easy to show that a given deterministic encoding is well-distributed: the character sum can be interpreted as an Artin character sum on the covering curve \( C \), which is bounded by \((2g_C + 2) \sqrt{q}\) according to a theorem by Weil (corollary of the Riemann hypothesis for curves).
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A simple encoding to hyperelliptic curves

• The first deterministic point-encoding function to hyperelliptic curves of a very special shape, \( y^2 = x^{2g+1} + ax + b \) was proposed by Ulas, as a generalization of the Shallue-van de Woestijne technique.

• More recently, Kammerer, Lercier and Renault proposed several Icart-like encoding functions to hyperelliptic curves of somewhat complicated but more general shape.

• We proposed a much simpler encoding function to the family of odd hyperelliptic curves \( H : y^2 = g(x) \) where \( g \) is an odd polynomial, over \( \mathbb{F}_q, \ q \equiv 3 \pmod{4} \). This encoding has many nice properties.

• Easy to describe: for any \( t \in \mathbb{F}_q \), one of \( g(t) \) or \( g(-t) \) is a square; define the point \( f(t) \) as \( y^2 = g(\pm t) \) accordingly, and set \( x \) such that \( f(-t) = -f(t) \).

• This encoding is very simple to compute, and is (almost) a bijection \( f : \mathbb{F}_q \rightarrow H(\mathbb{F}_q) \). In particular, it is admissible.
Encoding and hashing to the Jacobian

- The group used in hyperelliptic curve cryptography is the Jacobian $J$ of the curve: it is this group that we should seek to encode or hash to.
- Hashing at least is easy. All previously mentioned encodings to hyperelliptic curves $H$ are also well-distributed, in the sense that for all nontrivial characters $\chi$ of $J(\mathbb{F}_q)$:

\[
\left| \sum_{u \in \mathbb{F}_q} \chi(f(u)) \right| \leq B \sqrt{q}
\]

- Admissibility of $(u_1, \ldots, u_s) \mapsto f(u_1) + \cdots + f(u_s)$ again follows formally, as soon as $s$ is greater that the genus $g$ of $H$.
- Our encoding to odd hyperelliptic curves allows a different construction: an injective encoding to the Jacobian. Take $(u_1, \ldots, u_g) \mapsto f(u_1) + \cdots + f(u_g)$, from the set of tuples such that $u_1 < \cdots < u_g$ and $u_i + u_j \neq 0$. This is injective and reaches a fraction of about $1/g!$ points of $J(\mathbb{F}_q)$. Necessary for e.g. El Gamal encryption.
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Some open problems

- Encoding to some missing types of curves: Baretto-Naehrig elliptic curves, more hyperelliptic curves...

- Bounded leakage. It is easy to distinguish the output of the whole Icart's function from a uniform distribution. And the same is true with just the $x$-coordinate. However, if one only has the top half bits of $x$, the output is uniform. At which point between these two extremes can a distinguisher still work?

-Injective deterministic encodings: they are probably even more useful than hash functions, but have only been constructed on a few curves. Extend this to at least ordinary elliptic curves. A proper formalization of desired properties would be desirable.

- Impossibility results in generic groups.
Summary

- Hashing and encoding to (hyper)elliptic curves are problems worth looking into.
- Some good candidates are known, but there is still a lot of work to do.
- Plenty of nice problems, from pure mathematics to applied crypto.
Thank you!