Analysis of BKZ

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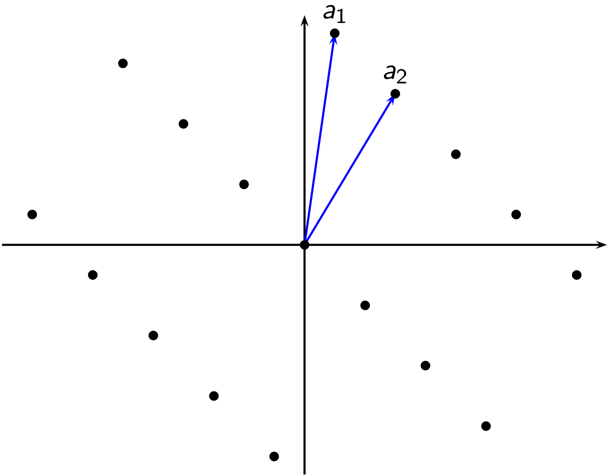
May 5, 2011

Lattices



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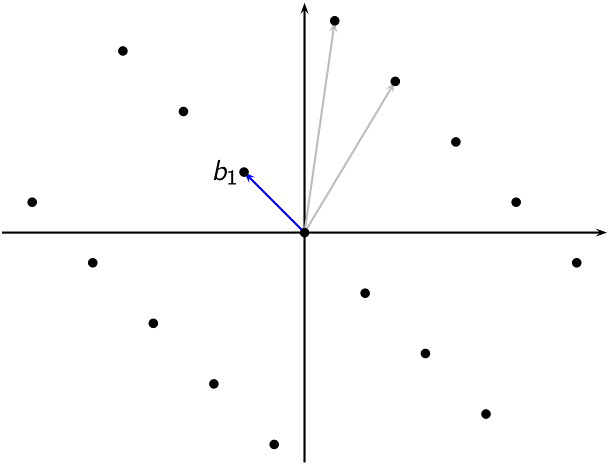
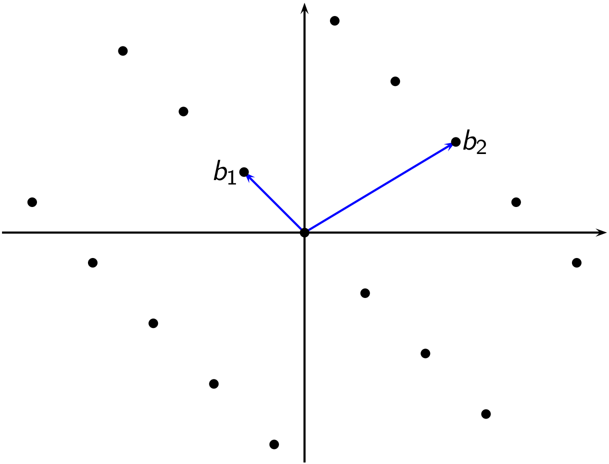


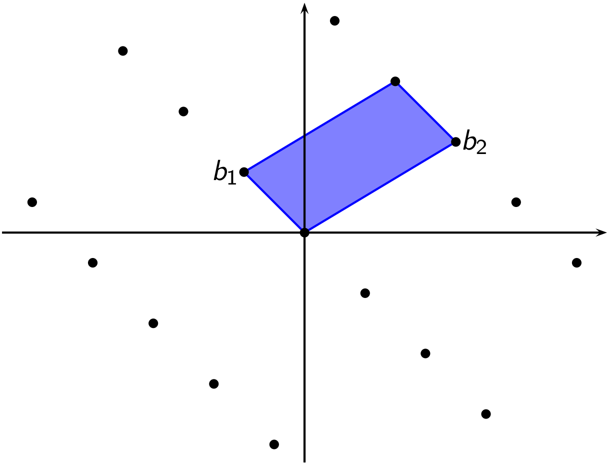
Shortest vector problem (SVP)

Lattices



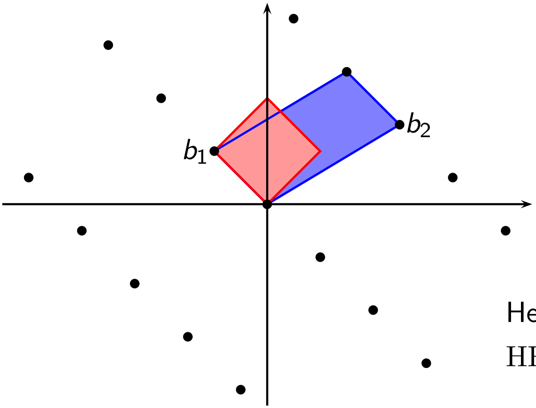
Lattice reduction

Lattices



Determinant

Lattices



Hermite factor:

$$\mathrm{HF}(b_1, \ldots, b_n) = \frac{\|b_1\|}{(\det L)^{1/n}}$$

- Goal of lattice reduction: find a basis with small HF.
- If b_1 is a shortest vector, then $\mathrm{HF}(b_1, \ldots, b_n) \leq \sqrt{\gamma_n}$, with γ_n = Hermite constant $\leq n$.

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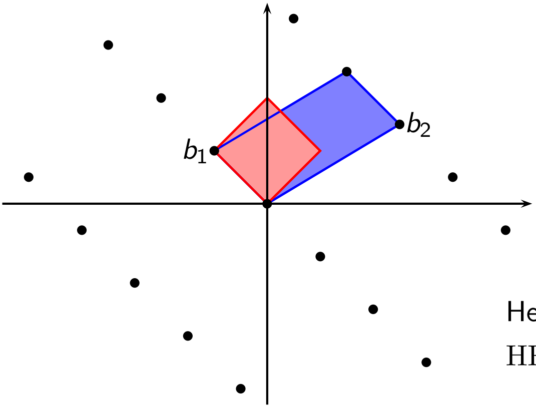

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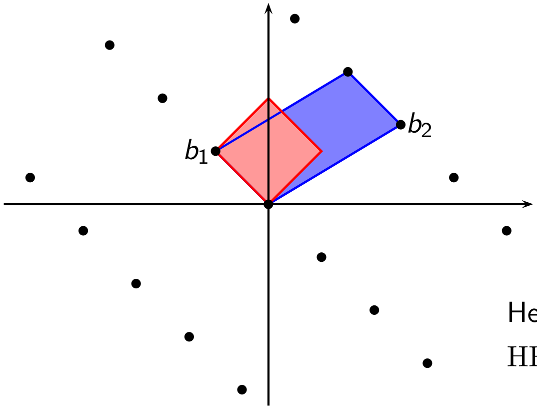

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Lattice reduction

Lattice reduction and shortest vector problem:

- The security of lattice-based cryptosystems relies on the hardness of (variants of) SVP.
- SVP and lattice reduction are interdependent problems.

Hierarchy of lattice reductions in dimension n:

	HKZ	BKZ$_\beta$	LLL
Hermite factor	$\sqrt{\gamma_n}$	$\simeq (\gamma_\beta(1+\epsilon))^{\frac{n-1}{2(\beta-1)}}$	$(\gamma_2(1+\epsilon))^{\frac{n-1}{2}}$
Time	$2^{O(n)}$	$2^{O(\beta)} \times ?$	Poly(n)

HKZ = Hermite-Korkine-Zolotareff

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History of BKZ

Practice

- Schnorr and Euchner (1994): algorithm for BKZ-reduction, without complexity analysis.
- Shoup: first public implementation of BKZ in NTL.
- Gama and Nguyen (2008): BKZ behaves badly when the block size is ≥ 25.

Theory

- Schnorr (1987): first hierarchies of algorithms between LLL and HKZ.
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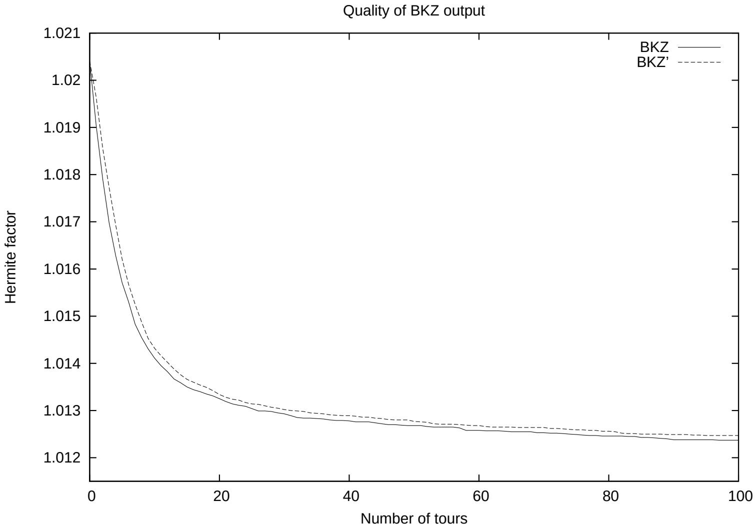
Slide-reduction:

- Outputs a basis whose theoretical quality is equivalent to BKZ.
- Polynomial number of calls to a SVP oracle.
- Not as efficient as BKZ in practice.

Progress made during the execution of BKZ



Experience on 64 LLL-reduced knapsack-like matrices ($n = 108, \beta = 24$).

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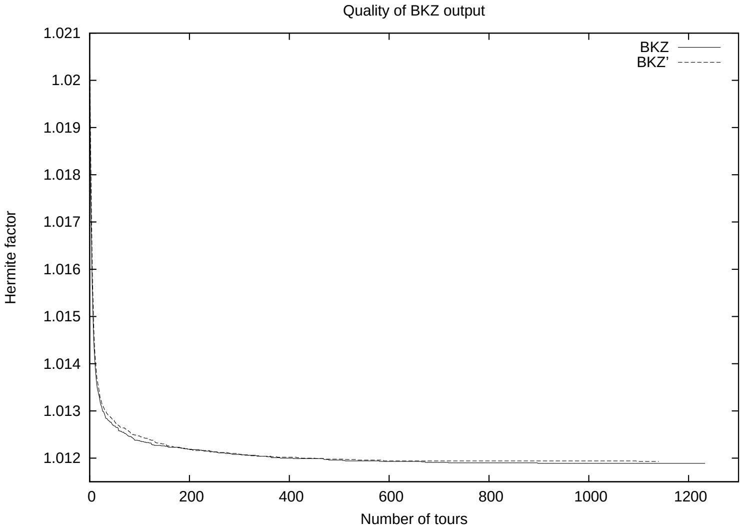

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Our result

γ_β = Hermite constant $\leq \beta$.
L a lattice with basis $(b_1, \ldots, b_n)$.

Theorem

After $\mathcal{O}\left(\frac{n^3}{\beta^2}\left(\log\frac{n}{\epsilon} + \log\log\max\frac{\|b_i\|}{(\det L)^{1/n}}\right)\right)$ *calls to* HKZ_β*, BKZ_β returns a basis C of L such that:*

$$\mathrm{HF}(C) \leq (1+\epsilon)\gamma_\beta^{\frac{n-1}{2(\beta-1)}+\frac{3}{2}}$$

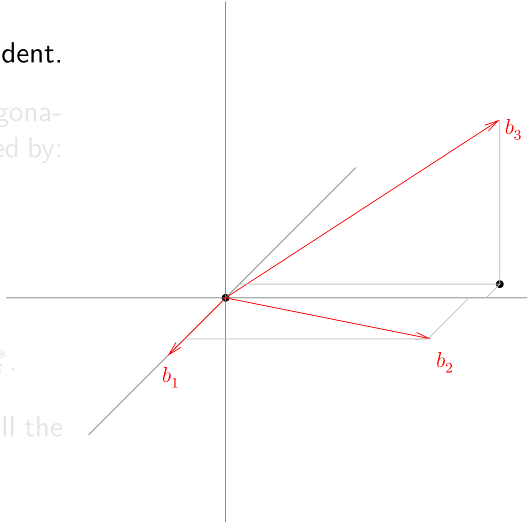
Gram-Schmidt orthogonalization

$b_1, \ldots, b_n$ linearly independent.

The Gram-Schmidt orthogonalization $b_1^*, \ldots, b_n^*$ is defined by:

- For all $i > j$, $\mu_{i,j} = \frac{(b_i, b_j^*)}{\|b_j^*\|^2}$.
- For all i, $b_i^* = b_i - \sum_{j<i} \mu_{i,j} b_j^*$.

A basis is size-reduced if all the $|\mu_{i,j}|$ are $\leq \frac{1}{2}$.



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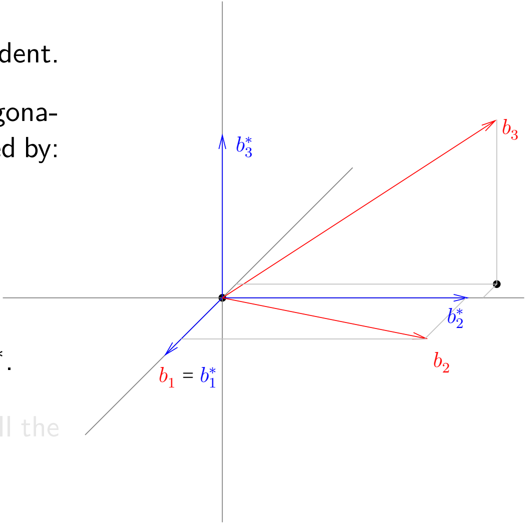

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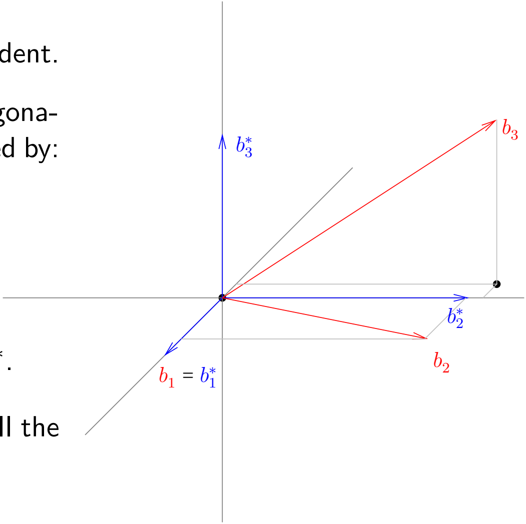

LLL

B is δ-LLL-reduced if:

- It is size-reduced;
- $\delta\|b_i^*\|^2 \leq \|b_{i+1}^*\|^2 + \mu_{i+1,i}^2\|b_i^*\|^2$ for all $i < n$.
  $\rightarrow x_i \leq \frac{1}{2}\log\gamma_2 + x_{i+1} - \log\delta \quad (x_i = \log\|b_i^*\|)$

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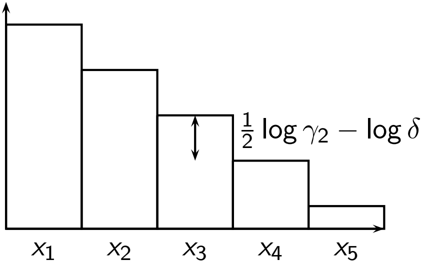

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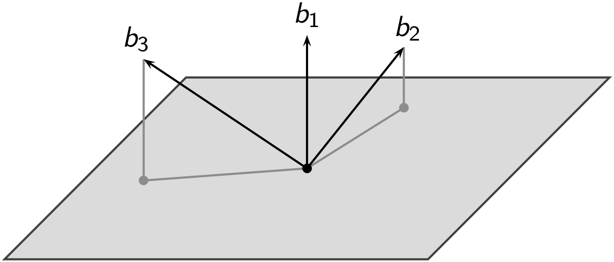

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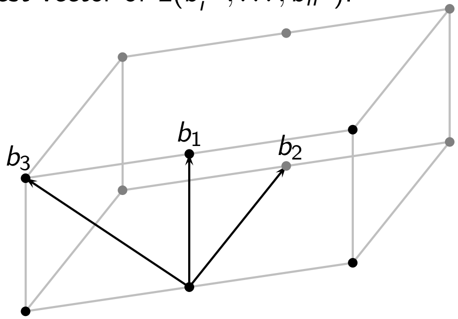

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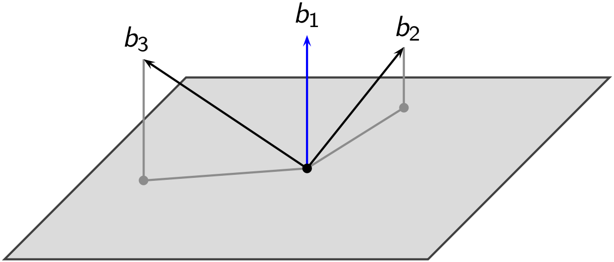

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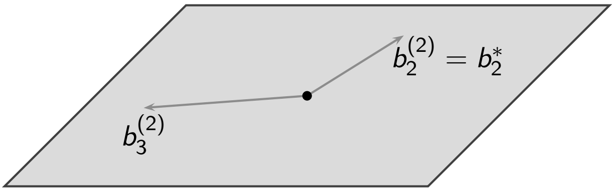

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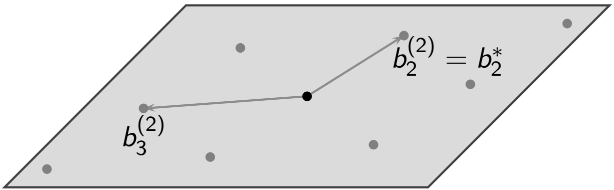

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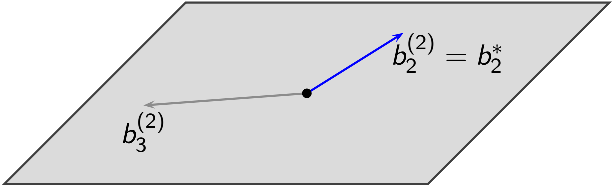

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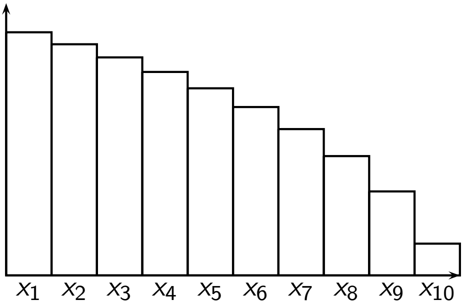
For $i < n$,
$\mathrm{HF}(b_i^{(i)}, \ldots, b_n^{(i)}) \leq \sqrt{\gamma_{n-i+1}}$

Worst-case HKZ profile:

$$\begin{aligned} x_i &= \log \|b_i^*\| \\ &= \mathcal{O}(\log^2(n-i)) \end{aligned}$$



BKZ

Algorithm (BKZ$_\beta$, modified version)

Input: B of dimension n.
Repeat ... times
    For i from 1 to $n - \beta + 1$ do
        Size-reduce B.
        HKZ-reduce the projected sublattice $(b_i^{(i)}, \ldots, b_{i+\beta-1}^{(i)})$.
        Report the transformation on B.

Termination?

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Sandpile model

- We consider only $x_i = \log \|b_i^*\|$ for $i \leq n$.
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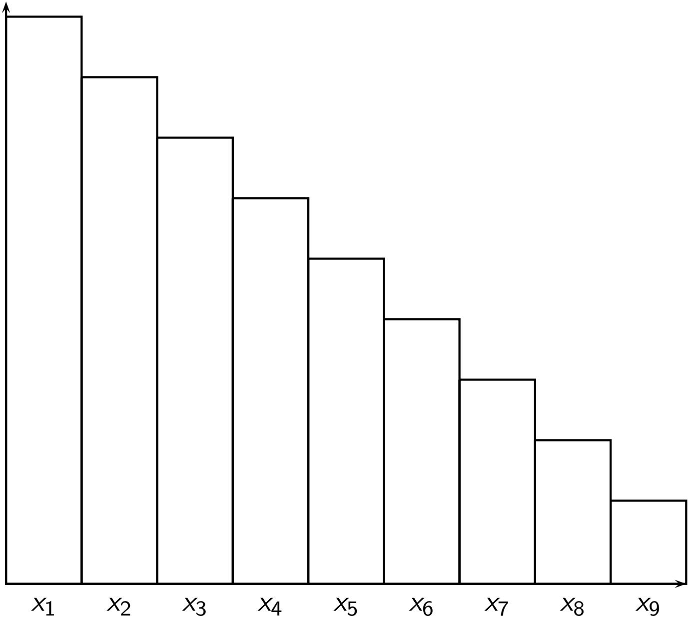
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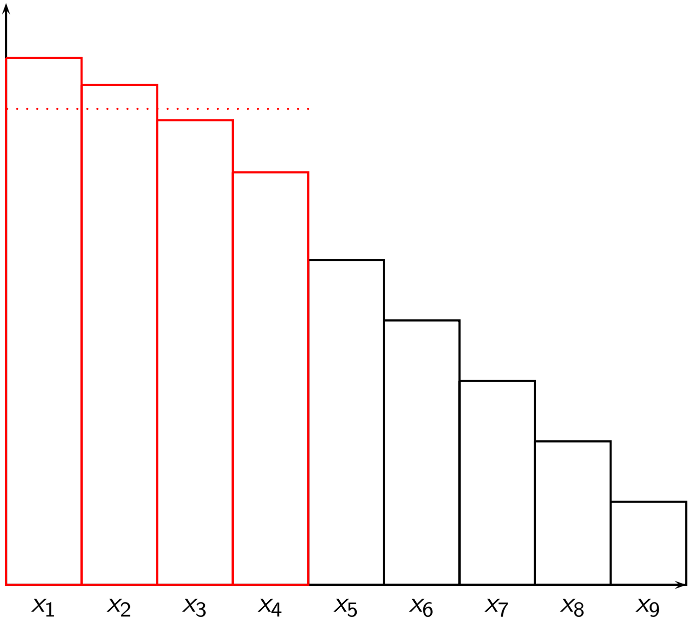
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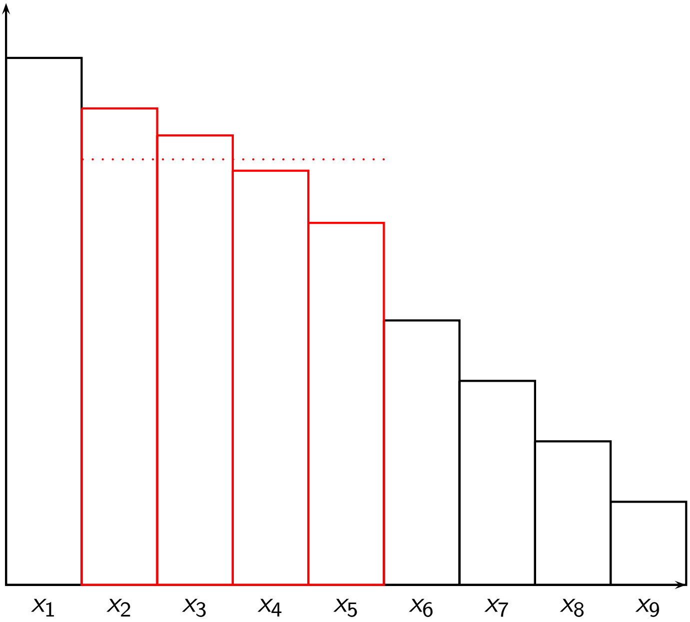
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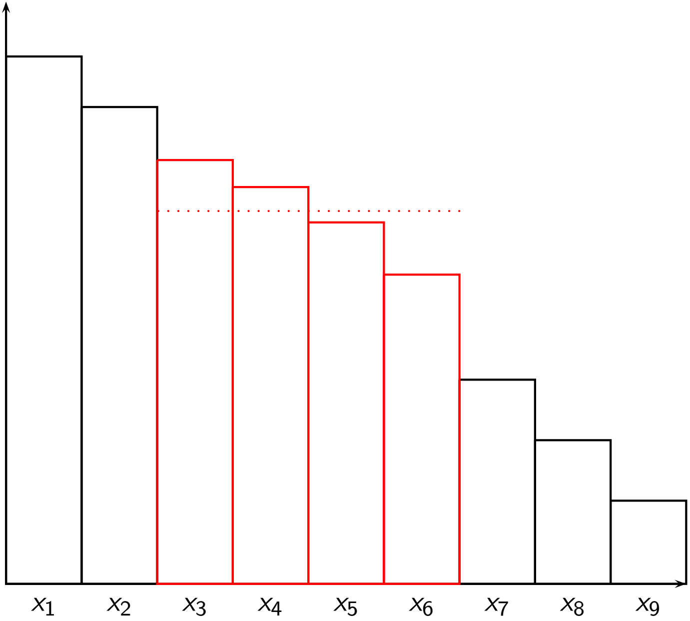

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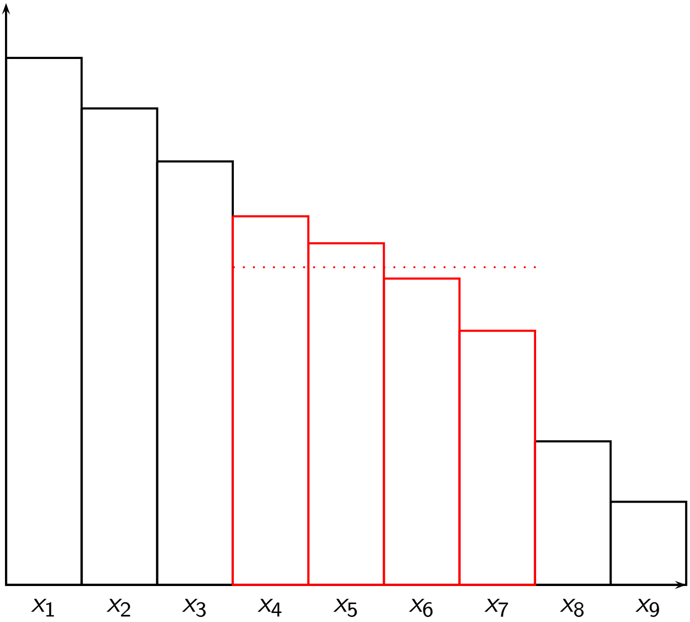

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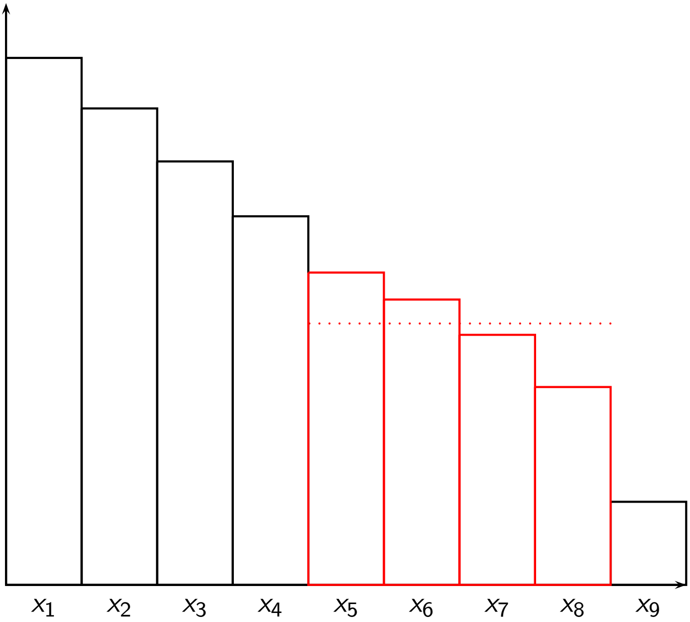
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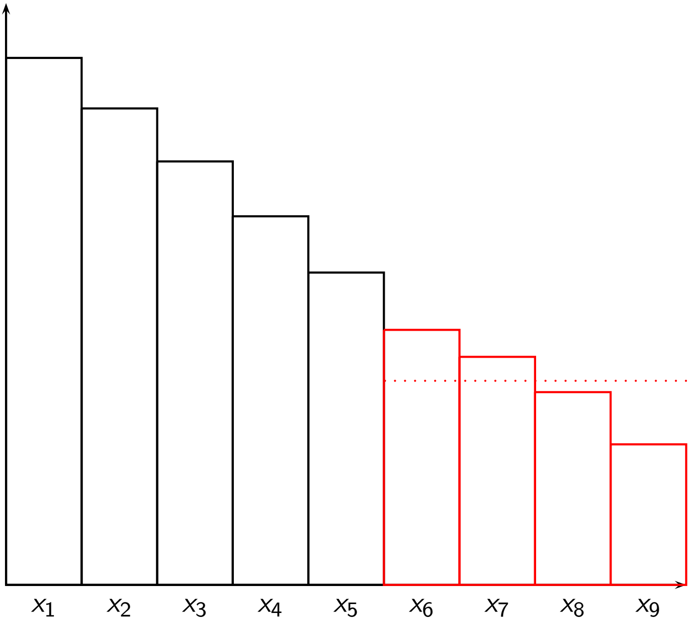
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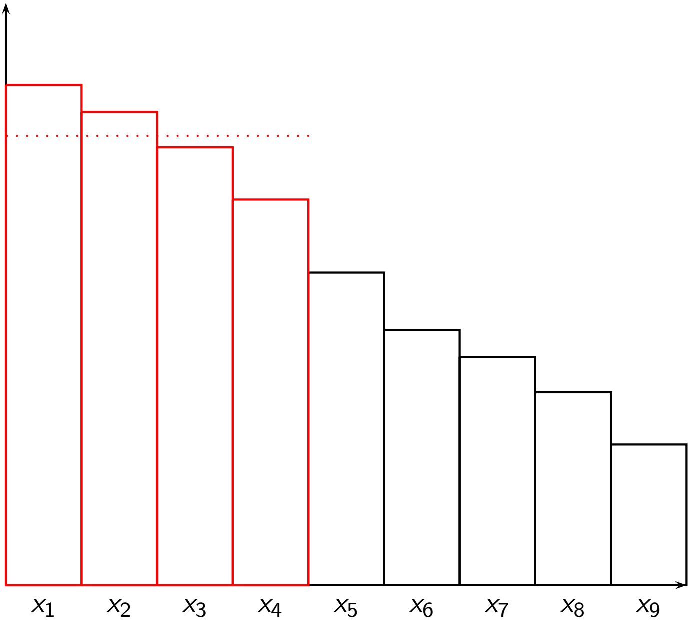
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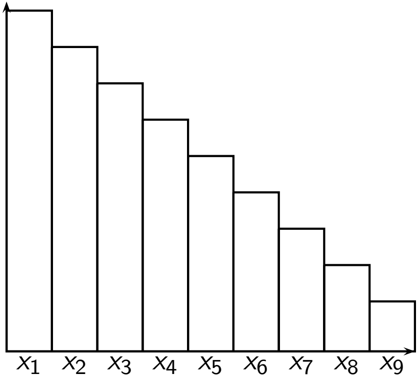
Matricial interpretation



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$X_2 \leftarrow A_2 X_1 + \Gamma_2$

...

$X_k = A_k X_k + \Gamma_k$

with $k = n - \beta + 1$

A full tour:

$X' \leftarrow AX + \Gamma$

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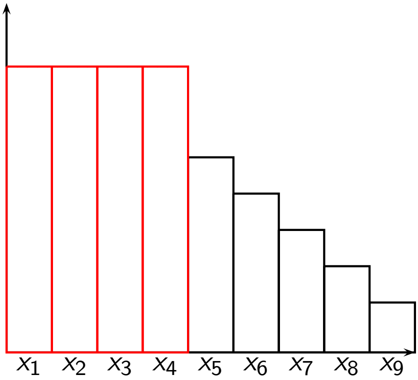

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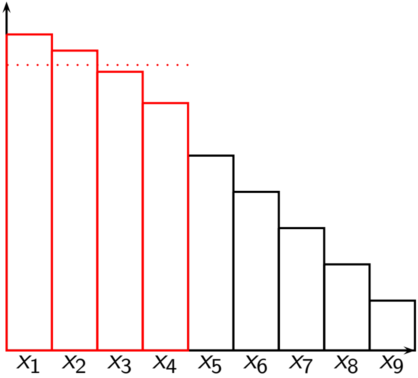

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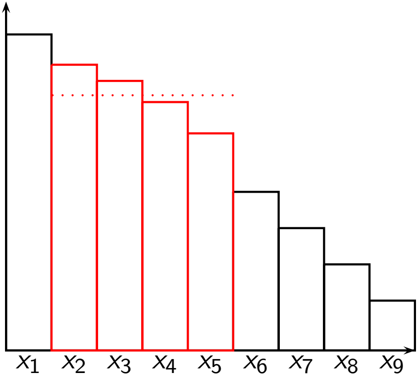

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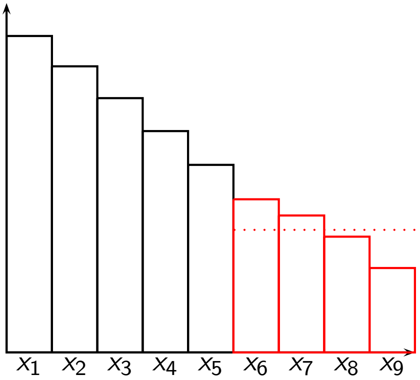

$X = (x_1, \ldots, x_n)^T$
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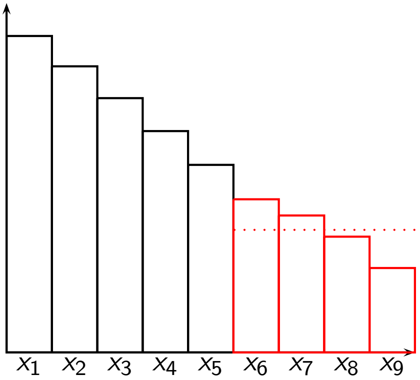

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Expected properties of the model

$$X \leftarrow AX + \Gamma$$

- Well-reduced output:
  $\rightarrow$ study of fixed points ($X^\infty = AX^\infty + \Gamma$).
- Convergence in a polynomial number of steps:
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$$x_i^\infty = \frac{\beta}{2(\beta-1)} \log \gamma_\beta + \sum_{j=i+1}^{i+\beta} \frac{x_j^\infty}{\beta - 1}.$$

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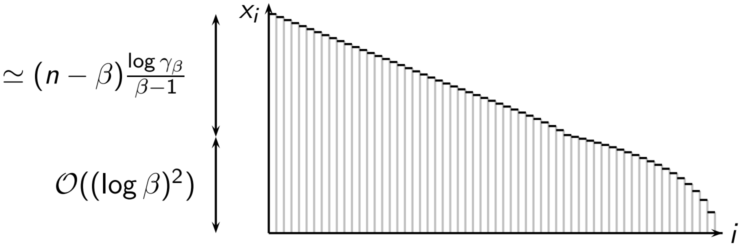

Eigenvalues of $A^T A$

- Method: study of the roots of the characteristic polynomial of $A^T A$.
- Let $\chi_n(\lambda) = \det(\lambda I_n - A_n^T A_n)$.
  Recurrence formula:

$$\chi_{n+2}(\lambda) = \frac{[2\beta(\beta-1)+1]\,\lambda - 1}{\beta^2}\chi_{n+1} - \left(\frac{\beta-1}{\beta}\right)^2 \lambda^2 \chi_n$$

- By a change of variable, it becomes a classical recurrence (Chebyshev polynomials):

$$\psi_{n+2}(\mu) = 2\mu\psi_{n+1}(\mu) - \psi_n(\mu)$$

(change of variable: $\tau(\mu) = 2\beta(\beta-1)(\mu-1)$ et $\psi_n(\mu) = \left(\frac{\beta}{\beta-1}\right)^{n-\beta} \cdot \frac{\bar{\chi}_n(1-\tau(\mu))}{\tau(\mu)}$)

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Results on the sandpile model

- The slope $-\frac{\log \gamma_\beta}{\beta-1}$ of the fixed point corresponds to a Hermite factor $\frac{\|b_1\|}{(\det L)^{1/n}}$ close to $\gamma_\beta^{\frac{n-1}{2(\beta-1)}}$.
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Comparison between the model and BKZ

When the determinant is fixed, there is no vector inequality on the x_i's between:

- a worst-case HKZ-reduced basis (equalities in Minkowski inequalities)
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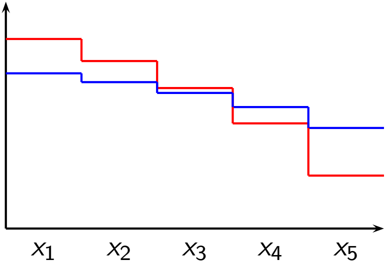

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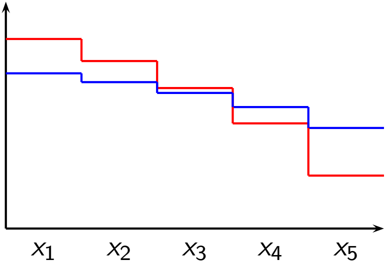

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Change of basis

- Obtaining information on the individual x_i's is difficult.
- The model can give some information on $\pi_i = \frac{1}{i}\sum_{j=1}^{i} x_j$, the mean of the first x_j's.
- New dynamical system: $\Pi \leftarrow \widetilde{A}\Pi + \widetilde{\Gamma} \quad (\widetilde{A} = PAP^{-1})$
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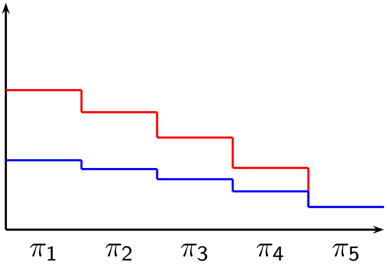

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Results on BKZ_β

Using the inequality $\Pi' \leq \widetilde{A}\Pi + \widetilde{\Gamma}$ recursively gives:

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The upper bound on the eigenvalues of $A^T A$ is used to bound the 2-norm of the right term.

$$\Pi^{[k]} - \Pi^{\infty} \leq (1 + \log n)^{\frac{1}{2}} \left(1 - \frac{\beta^2}{2n^2}\right)^{\frac{k}{2}} \|\Pi^{[0]} - \Pi^{\infty}\|_2$$

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- Swaps in LLL / HKZ_2 = Gauss-reductions in BKZ_2.
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Quasi-linear LLL

In BKZ_2:

- Each Gauss-reduction costs $\widetilde{\mathcal{O}}(\log \max \|b_i\|)$.
- $\mathcal{P}oly(n) \times \log \log \max_i \frac{\|b_i^*\|}{(\det L)^{1/n}}$ Gauss-reductions.
- A basis such that $\frac{\|b_1\|}{(\det L)^{1/n}} \leq \sqrt{\frac{4}{3}}^{n-1}(1+\epsilon)$ is returned.
- With more work, it is possible to obtain an LLL-reduced basis.

Conclusion

- The optimal quality that can be proven for BKZ_β is reached in a polynomial number of calls to HKZ_β.
- Binary complexity of BKZ_2?
- Adaptive strategies.
- In practice, the algorithm reaches better approximation factors than expected.
  $\rightarrow$ For how long is it interesting to continue the execution once we go beyond the theorical factor?