The complete generating function for Gessel walks is algebraic

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joint work with

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Context: nearest-neighbour walks in $\mathbb{N}^2$

- **Set of admissible steps** $\mathcal{S} \subseteq \{\nearrow, \leftarrow, \nwarrow, \uparrow, \rightarrow, \searrow, \downarrow\}$.
- **$\mathcal{S}$-walks** = walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in $\mathcal{S}$.
- $f(n; i, j) =$ number of $\mathcal{S}$-walks ending at $(i, j)$ and consisting of exactly $n$ steps. Complete generating function

$$F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f(n; i, j)x^iy^j \right)t^n \in \mathbb{Q}[x, y][[t]].$$

**Questions:** Starting from $\mathcal{S}$, what can be said about $F(t; x, y)$? Is it **algebraic**, or **holonomic transcendental**, or **non-holonomic**?

$F(t; 1, 1) \leadsto$ number of walks with prescribed number of steps;  
$F(t; 0, 0) \leadsto$ number of walks returning to the origin (excursions);  
$F(t; 1, 0) \leadsto$ number of walks ending on the horizontal axis.
Examples: **Kreweras and Gessel walks**

Kreweras walks: $S$-walks with $S = \{\downarrow, \leftarrow, \rightarrow\}$

Gessel walks: $S$-walks with $S = \{\uparrow, \nearrow, \leftarrow, \rightarrow\}$

Example: A Kreweras excursion of length 24.
Main results

Theorem (Kreweras 1965; 100 pages combinatorial proof!)

\[ K(t; 0, 0) = \left(1/3 \atop 3/2 \atop 2 \right)_{1} 27 t^{3} = \sum_{n=0}^{\infty} \frac{4^n (3^n)}{(n + 1)(2n + 1)} t^{3n} \]

Theorem (Gessel’s conjecture; Kauers, Koutschan, Zeilberger 2008)

\[ G(t; 0, 0) = \left(5/6 \atop 5/3 \atop 2 \right)_{1} 16 t^{2} = \sum_{n=0}^{\infty} \frac{(5/6)^n (1/2)^n (2)}{(5/3)^n (2)^n} (4t)^{2n} \]

Question: What about \( K(t; x, y) \) and \( G(t; x, y) \)?

Theorem (Bousquet-Mélou 2005) \( K(t; x, y) \) is algebraic.

Theorem (B. & Kauers 2008) \( G(t; x, y) \) is algebraic.

In particular, \( g(n; i, j) \) is holonomic for any pair \((i, j) \in \mathbb{N}^2\).

→ Effective, computer-driven, discovery and proof.
Methodology

Experimental mathematics approach:

(S1) high order expansions of generating power series;
(S2) guessing differential and/or algebraic equations they satisfy;
(S3) empirical certification of the guessed equations (sieving by inspection of their analytic, algebraic, arithmetic properties);
(S4) rigorous proof, based on (exact) polynomial computations.
Step (S1): high order series expansions

$f(n; i, j)$ satisfies the recurrence with constant coefficients

$$f(n + 1; i, j) = \sum_{(u, v) \in S} f(n; i - u, j - v) \quad \text{for} \quad n, i, j \geq 0$$

+ init. cond. $f(0; i, j) = \delta_{0,i,j}$ and $f(n; -1, j) = f(n; i, -1) = 0$.

Example: for the Kreweras walks,

$$k(n; i, j) = k(n - 1; i + 1, j) + k(n - 1; i, j + 1) + k(n - 1; i - 1, j - 1)$$

The recurrence is used to compute $F(t; x, y) \mod t^N$ for large $N$.

$$K(t; x, y) = 1 + xyt + (x^2y^2 + y + x)t^2 + (x^3y^3 + 2xy^2 + 2x^2y + 2)t^3$$
$$+ (x^4y^4 + 3x^2y^3 + 3x^3y^2 + 2y^2 + 6xy + 2x^2)t^4$$
$$+ (x^5y^5 + 4x^3y^4 + 4x^4y^3 + 5xy^3 + 12x^2y^2 + 5x^3y + 8y + 8x)t^5 + \cdots$$
Step (S2): guessing equations for $F(t; x, y)$, a first idea

In terms of generating series, the recurrence on $k(n; i,j)$ reads

$$
(xy - t(x + y + x^2y^2))K(t; x, y) = xy - tx K(t; x, 0) - ty K(t; 0, y)
$$

(KerEq)

- This kernel equation can be seen as a multivariate analogue of
  \[(1 - t - t^2) \cdot \sum_{n \geq 0} f_n t^n = 1,\] where $f_n$ are the Fibonacci numbers.

- A similar kernel equation holds for $F(t; x, y)$, for any $\mathcal{S}$-walk.

Corollary. $F(t; x, y)$ is holonomic (resp. algebraic) if and only if $F(t; x, 0)$ and $F(t; 0, y)$ are both holonomic (resp. algebraic).

- This simplification is crucial: equations for $G(t; x, y)$ are huge.
Step (S2): guessing equations for $F(t; x, 0)$ and $F(t; 0, y)$

**Task 1:** Given the first $N$ terms of $S = F(t; x, 0) \in \mathbb{Q}[x][[t]]$, search for a **differential equation** satisfied by $S$ at precision $N$:

$$L_{x,0}(S) = c_r(x, t) \frac{d^r S}{dt^r} + \cdots + c_1(x, t) \frac{dS}{dt} + c_0(x, t) \cdot S = 0 \text{ mod } t^N.$$ 

**Task 2:** Search for an **algebraic equation** $P_{x,0}(S) = 0 \text{ mod } t^N$.

- Both tasks amount to **linear algebra** in size $N$ over $\mathbb{Q}(x)$.
- In practice, we use **modular** Hermite-Padé approximation (Beckermann-Labahn algorithm) combined with (rational) evaluation-interpolation and rational number reconstruction.
- (Right) **gcds** of several candidates provide minimal equations.
Step (S2): guessing equations for $K(t; x, 0)$

The guessed operator of order 4 in $D_t = \frac{d}{dt}$, degree $(14, 11)$ in $(t, x)$

$$\mathcal{L}_{x,0} = t^3 \cdot (3t - 1) \cdot (9t^2 + 3t + 1) \cdot (3t^2 + 24t^2x^3 - 3xt - 2x^2) \cdot$$
$$\cdot (16t^2x^5 + 4x^4 - 72t^4x^3 - 18x^3t + 5t^2x^2 + 18xt^3 - 9t^4) \cdot$$
$$\cdot (4t^2x^3 - t^2 + 2xt - x^2) \cdot D_t^4 + \cdots$$

is such that $\mathcal{L}_{x,0}(K(t; x, 0)) = 0 \text{ mod } t^{100}$.

The guessed polynomial of tridegree $(6, 10, 6)$ in $(T, t, x)$

$$\mathcal{P}_{x,0} = x^6t^{10}T^6 - 3x^4t^8(x - 2t)T^5 +$$
$$+ x^2t^6 \left(12t^2 + 3t^2x^3 - 12xt + \frac{7}{2}x^2\right)T^4 + \cdots$$

is such that $\mathcal{P}_{x,0}(K(t; x, 0), t, x) = 0 \text{ mod } t^{100}$. 
Step (S2): guessing equations for $G(t; x, 0)$ and $G(t; 0, y)$

For Gessel walks, using $N = 1000$ terms of $G(t; x, y)$, we guessed

- $L_{x,0}$: order 11 in $D_t$, bidegree $(96, 78)$ in $(t, x)$, 61 digits coeffs
- $L_{0,y}$: order 11 in $D_t$, bidegree $(68, 28)$ in $(t, y)$, 51 digits coeffs

such that $L_{x,0}(G(t; x, 0)) = L_{0,y}(G(t; 0, y)) = 0 \mod t^{1000}$.

- For a fixed value $x_0$, and modulo a prime $p$, many (non-minimal) operators in $\mathbb{Z}_p[t][D_t]$ for $G(t; x_0, 0)$ can be guessed by Hermite-Padé.

- Still: reconstructing from one of them an operator in $\mathbb{Q}[t, x][D_t]$ for $G(t; x, 0)$ is too costly.

- However, the reconstruction (wrt $x$) is feasible if applied to the minimal-order operator $= \gcd$.

- Guessing $L_{x,0}$ by undetermined coefficients would have required solving a dense linear system $91956 \times 91956$ with $\approx 1000$ digits entries!
Step (S2): guessing differential equations for $G(t; x, y)$?

Feasible in principle: kernel equation + closure by differential lclm.

- Obstacle: this lclm has order 20 in $D_t$, tridegree $(359, 717, 279)$ in $(t, x, y) \rightarrow 1.5$ billion integer coefficients (!)

- Thus: at this point, we had guesses for differential equations for $G(t; x, 0)$ and $G(t; 0, y)$, but no proof that they are correct and no hope to compute a candidate differential equation for $G(t; x, y)$.

▷ Remember: it was believed (e.g. by Gessel and Zeilberger, who popularized the problem) that $G(t; x, y)$ is not algebraic.

▷ This explains why no one (including us) tried – at this stage – to search for algebraic equations. Worse: no one even remarked that Gessel’s expression $\sum_{n=0}^{\infty} \frac{\alpha \beta}{n!} (t^n)$ for excursions is algebraic.
Step (S3): empirical certification of guesses

Provide convincing evidence that the candidate $\mathcal{L}_{x,0}$ is correct:

1. Size sieve: $\mathcal{L}_{x,0}$ has reasonable bit size compared to an artefact solution of the Hermite-Padé approximation problem.

2. Algebraic sieve: High order series matching. $\mathcal{L}_{x,0}$ verifies $\mathcal{L}_{x,0}(F(t; x, 0)) = 0 \mod t^{N+\varepsilon}$.

3. Analytic sieve: singularity analysis. $\mathcal{L}_{x,0}$ is Fuchsian (all of its singular points are regular singular).

4. Arithmetic sieve: $\mathcal{L}_{x,0}$ is globally nilpotent (see below).
Step (S3): $G$-series and global nilpotence

**Def.** A power series $\sum_{n \geq 0} \frac{a_n}{b_n} t^n$ in $\mathbb{Q}[[t]]$ is called a *$G$-series* if it is (a) holonomic; (b) analytic at $t=0$; (c) $\exists C > 0, \text{lcm}(b_0, \ldots, b_n) \leq C^n$.

**Examples:** $_{2}F_{1} \left( \begin{array}{c} \alpha, \beta \\ \gamma \end{array} \right| t)$, $\alpha, \beta, \gamma \in \mathbb{Q}$; algebraic functions (Eisenstein).

**Thm. (Chudnovsky 1985)** The minimal-order differential operator annihilating a $G$-series is *globally nilpotent*: for almost all prime numbers $p$, it right-divides $D_t^{p\mu}$ modulo $p$, for some $\mu \in \mathbb{N}$.

**Examples:** $t(1-t)D_t^2 + (\gamma - (\alpha + \beta + 1)t)D_t - \alpha\beta t$; algebraic resolvents.

**Thm. (B. & Kauers)** If $F(t; x, 0)$ is holonomic, then it’s a $G$-series.

▷ The guessed operators for $K(t; x, 0)$, $G(t; x, 0)$, $G(t; 0, y)$ pass this arithmetic test: they are all globally nilpotent.

▷ And, unexpectedly, even more...
Step (S3): Grothendieck’s conjecture and the big surprise

Conjecture (Grothendieck) $\mathcal{L}(S) = 0$ possesses a basis of algebraic solutions if and only if $\mathcal{L}$ globally nilpotent with $\mu = 1$.

▷ Big surprise: the guessed operators for $G(t; x, 0)$ and $G(t; 0, y)$ share this property for $5 \leq p < 100 \Rightarrow$ this strongly indicates that $G(t; x, 0)$ and $G(t; 0, y)$, and thus $G(t; x, y)$, should be algebraic!

Once we suspect algebraicity of $G(t; x, 0)$ and $G(t; 0, y)$, we guess candidates for annihilating polynomials

- $P_{x,0}$ in $\mathbb{Z}[x, t, T]$ of tridegree $(32, 43, 24)$ in $(x, t, T)$, 21 digits
- $P_{0,y}$ in $\mathbb{Z}[y, t, T]$ of tridegree $(40, 44, 24)$ in $(y, t, T)$, 23 digits

such that

$$P_{x,0}(x, t, G(t; x, 0)) = P_{0,y}(x, t, G(t; 0, y)) = 0 \mod t^{1200}.$$
Step (S4): warm-up – Gessel excursions

Theorem \( G(t; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} (4t)^{2n} \) is algebraic.

Proof 1: This \( _3F_2 \) series is a \( _2F_1 \) series in disguise:

\[
_3F_2\left(\begin{array}{ccc}
\frac{5}{6} & 1/2 & 1 \\
\frac{5}{3} & 2 & \end{array} \mid 16t^2\right) = \frac{1}{t^2} \left( \frac{1}{2} _2F_1\left(\begin{array}{ccc}
-1/6 & -1/2 & \\
2/3 & & \end{array} \mid 16t^2\right) - \frac{1}{2} \right).
\]

Schwarz’s classification of algebraic \( _2F_1 \)'s allows to conclude.

Proof 2: \textit{Guess} a polynomial \( P(T, t) \) in \( \mathbb{Q}[T, t] \), then prove that \( P \) admits the power series \( g(t) = G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} g_n t^n \) as a root.

1. Such a \( P \) can be \textit{guessed} from the first 100 terms of \( g(t) \).
2. Implicit function theorem: \( \exists! \) root \( r(t) \in \mathbb{Q}[[t]] \) of \( P \).
3. \( r(t) = \sum_{n=0}^{\infty} r_n t^n \) being algebraic, it is holonomic, and so is \( (r_n) \):

\[
(n + 2)(3n + 5)r_{n+1} - 4(6n + 5)(2n + 1)r_n = 0, \quad r_0 = 1.
\]

\( \Rightarrow \) solution \( r_n = \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} 4^{2n} = g_n \), thus \( g(t) = r(t) \) is algebraic.
Step (S4): rigorous proof for Kreweras walks

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \ldots$
   in the kernel equation
   \[
   (xy - (x + y + x^2y^2)t)K(t; x, y) = -xy + xtK(t; x, 0) + ytK(t; 0, y)
   \]
   \[
   \triangleright 0
   \]
   shows that $U = K(t; x, 0)$ satisfies the reduced kernel equation
   \[
   x \cdot y_0 - x \cdot t \cdot U(t, x) = y_0 \cdot t \cdot U(t, y_0)
   \]
   (R Ker Eq)

2. $U = K(t; x, 0)$ is the unique solution in $\mathbb{Q}[[x, t]]$ of (R Ker Eq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t, x)$ in $\mathbb{Q}[[x, t]]$.

4. Resultant computations + verification of initial terms
   \[
   \implies U = H(t, x) \text{ also satisfies (R Ker Eq)}.
   \]

5. Uniqueness: $H(t, x) = K(t; x, 0) \implies K(t; x, 0)$ is algebraic!
Algebraicity of Kreweras walks: our Maple proof in a nutshell

[bostan@venus ~]$ maple

\|/~| Maple 11 (X86 64 LINUX)
._|\|_/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2007
\ MAPLE / All rights reserved. Maple is a trademark of
<____ ____> Waterloo Maple Inc.
           | Type ? for help.

# HIGH ORDER EXPANSION (S1)
> st, bu := time(), kernelopts(bytesused):
> f := proc(n, i, j)
    option remember;
    if i < 0 or j < 0 or n < 0 then 0
    elif n = 0 then if i = 0 and j = 0 then 1 else 0 fi
    else f(n-1, i-1, j-1) + f(n-1, i, j+1) + f(n-1, i+1, j) fi
end:
> S := series(add(add(f(k, i, 0)*x^i, i = 0..k)*t^k, k = 0..80), t, 80):

# GUESSING (S2)
> libname := ".", libname: gfun:-version();
> gfun:-seriestoalgeq(S, Fx(t)):
> P := collect(numer(subs(Fx(t) = T, %[1])), T):

# RIGOROUS PROOF (S4)
> ker := (T, t, x) -> (x + T + x^2*T^2)*t - x*T:
> pol := unapply(P, T, t, x):
> p1 := resultant(pol(z - T, t, x), ker(t*z, t, x), z):
> p2 := subs(T = x*T, resultant(numer(pol(T/z, t, z)), ker(z, t, x), z)):
> normal(primpart(p1, T)/primpart(p2, T));

# time (in sec) and memory consumption (in Mb)
> trunc(time()-st), trunc((kernelopts(bytesused)-bu)/1000^2);
15, 83
Step (S4): rigorous proof for Gessel walks

Two difficulties: $G(t; x, y) \neq G(t; y, x)$ and $G(t; 0, 0)$ occurs in (KerEq)

\[
poly(x, y, t)G(t; x, y) = xy + tG(t; 0, 0) - (1 + y)tG(t; 0, y) - tG(t; x, 0)
\]

\[
\equiv 0 \implies y_0(t, x) = 0 + \frac{1}{x} t + \frac{x^2 + 1}{x^2} t^2 + \frac{x^4 + 3x^2 + 1}{x^3} t^3 + \ldots
\]

or $x_0(t, y) = 0 + \frac{y + 1}{y} t + \frac{(y + 1)^3}{y^2} t^3 + \frac{2(y + 1)^5}{y^3} t^5 + \ldots$

This gives two equations connecting $G(t; x, 0)$ and $G(t; 0, y)$:

\[
G(t; x, 0) = xy_0/t + G(t; 0, 0) - (1 + y_0)G(t; 0, y_0)
\]

\[
(1 + y)G(t; 0, y) = yx_0/t + G(t; 0, 0) - G(t; x_0, 0)
\]

For fixed $G(t; 0, 0)$, they uniquely define $G(t; x, 0)$ and $G(t; 0, y)$.

- Resultant size: $\deg_T = 48$, $\deg_t = 90$, $\deg_y = 64$, digits = 47

$\implies$ fast algorithms needed (B., Flajolet, Salvy & Schost 2006).
## Conclusion

1. **Guess’n’Prove approach** based on modern CA algorithms.
2. Brute-force approach and/or use of naive algorithms = hopeless. E.g. size of algebraic equations for $G(t; x, y) \approx 30\text{Gb}$.
3. Going further: **experimental classification** of 2D and 3D walks: \((B. \ & Kauers \ FPSAC’09) \rightarrow 3500 \text{ cases treated}; \approx 4\% \text{ holonomic.}\) Matches the results of Bousquet-Mélou and Mishna (2D).
4. **Remarkable properties discovered experimentally**: explanation?
   4.1 **algebraic cases**: solvable Galois groups + genus 0, 1 and 5(!)

\[
G(t; 1, 1) = -\frac{3}{6t} + \frac{\sqrt{3}}{6t} \sqrt{U(t) + \sqrt{\frac{16t(2t + 3) + 2}{(1 - 4t)^2 U(t)}} - U(t)^2 + 3}
\]

where $U(t) = \sqrt{1 + 4t^{1/3}(4t + 1)^{1/3}/(4t - 1)^{4/3}}$.

4.2 **transcendental holonomic**: operators factor as $L_2(2) \cdot L_1(1) \cdots L_t(1)$ $\rightarrow$ iterated integrals of $2F_1$’s (cf. Dwork’s conjecture)

\[
F_{\ldots\cdot\cdot}(t; 0, 0) = \frac{2}{t^2} \int_0^t \tau(1 - 12\tau^2)(4\tau^2 + 1) \cdot _2F_1\left(\begin{array}{c}5/4, 7/4 \\ 2 \end{array}\middle| \frac{64\tau^4}{(1 - 4\tau^2)^2}\right) d\tau.
\]
Theorem Let \( V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots \) be the root of
\[
256V^3t^2 - (V - 1)(V + 3)^3 = 0,
\]
let \( U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \cdots \) be the root of
\[
 x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\
- xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0,
\]
let \( W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \cdots \) be the root of
\[
y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.
\]

Then \( G(t; x, y) \) is equal to
\[
\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2} - \frac{1}{tx(y+1)}.
\]
**Bonus: existence of the series root of** \( P = P_{x,0} \)

1. **Question:** Prove the existence of a root \( H \in \mathbb{Q}[[x, t]] \) of \( P \).
2. **Difficulty:** \( P(1, 0, 0) = \frac{\partial P}{\partial t}(1, 0, 0) = 0 \rightarrow \text{IFT does not apply.} \)
3. **Workaround:** exploit zero genus \( \rightarrow \) there exist \( R_1 \) and \( R_2 \)

\[
R_1(U, x) = \frac{(U^4 x^2 + 2U^2(U + 1)^2 x + 1 + 4U + 6U^2 + 2U^3 - U^4) h(U, x)}{(1 + U)^2(1 + 2U + U^2 + U^2 x)^4},
\]
\[
R_2(U, x) = \frac{U(1 + U)(1 + 2U + U^2 + U^2 x)^2}{h(U, x)},
\]

with \( h \in \mathbb{Q}[x, U] \) such that:

- \( P(R_1(U, x), R_2(U, x), x) = 0; \)
- \( \text{IFT applies to } R_2 - t: \) there exists a power series in \( \mathbb{Q}[[x, t]] \)

\[
U_0(t, x) = t + t^2 + (x+1)t^3 + (2x+5)t^4 + (2x^2 + 3x + 9)t^5 + ... 
\]

such that \( R_2(U_0, x) = t. \)

\[
\rightarrow P\left( R_1(U_0, x), t, x \right) = P\left( R_1(U_0, x), R_2(U_0, x), x \right) = 0. 
\]
Experimental classification of 2D walks with holonomic $F(t; 1, 1)$

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Equation sizes = \{order, degree\}(rec, diffeq, algeq).
Experimental classification of 2D walks with holonomic $F(t; 1, 1)$

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### Experimental classification of holonomic transcendental 3D walks

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**Equation sizes** = \{order, degree\} (rec, diffeq).